SUBFIELDS OF AMPLE FIELDS I. RATIONAL MAPS AND DEFINABILITY

ARNO FEHM

ABSTRACT. Pop proved that a smooth curve C over an ample field K with $C(K) \neq \emptyset$ has $|K|$ many rational points. We strengthen this result by showing that there are $|K|$ many rational points that do not lie in a given proper subfield, even after applying a rational map. As a consequence we gain insight into the structure of existentially definable subsets of ample fields. In particular, we prove that a perfect ample field has no existentially definable proper infinite subfields.

INTRODUCTION

Recall that a field K is called *ample* (or *large* after [Pop96]) if every smooth curve C defined over K with a K-rational point has infinitely many of them. The class of ample fields subsumes several seemingly unrelated classes of fields – separably closed fields and pseudo algebraically closed fields, real closed fields, and Henselian valued fields.

By introducing ample fields, Pop was able to reprove and generalise a couple of deep results on a certain important Galois theoretic conjecture of Dèbes and Deschamps (a vast generalization of the classical inverse Galois problem, see [DD99, §2.1.2]). Furthermore, all fields for which this conjecture has been proven so far turned out to be ample. Therefore, the class of ample fields now plays a decisive role in the study of general Galois theory. Moreover, in recent years ample fields drew attention in several other branches of mathematics – for example in the study of rationally connected varieties and torsors, see [Kol99], [MB01], in the study of Abelian varieties, see [Kob06], [LR08], [FP08], and in the study of definability in fields, see [Koe02], [PP07], [JK08]. We refer the reader to the introduction of [Pop08] for a more extensive survey of ample fields.

Harbater and Stevenson called a field K *very large* if every smooth curve C defined over K with a K-rational point has $|K|$ many such points. They proved that the absolute Galois group of a function field of one variable over a very large field K is a so-called quasi-free group, and they asked if the same holds for every ample field K (see [HS05]). Pop gave a positive answer to this question, by showing that every ample field is actually very large, cf. [Har06, Proposition 3.3].

In this work, we considerably strengthen this result by proving that in the situation above, C has $|K|$ many rational points *with respect to any proper subfield* of K. That is:

Theorem 1. *Let K be an ample field, C a curve defined over K with a simple K-rational point and $\varphi : C \to C'$ a separable dominant K-rational map to an affine curve $C' \subseteq \mathbb{A}^n$ defined over K. Then for every proper subfield K_0 of K,*

$$|\varphi(C(K)) \smallsetminus \mathbb{A}^n(K_0)| = |K|.$$

The proof of Theorem 1 is carried out in Section 1. It is based on a careful analysis of the trick of Koenigsmann that is used in Pop's proof.

Section 2 gives a model theoretic reinterpretation of Theorem 1: Motivated by what is known about definable subsets of some well-studied 'classical' ample fields, we apply Theorem 1 to investigate the structure of existentially definable (i.e. diophantine) subsets of ample fields. For example we prove that a perfect ample field has no existentially definable proper infinite subfields, and we point out connections with the recent work [JK08].

Notation

All varieties are geometrically irreducible and geometrically reduced. For any set X, the cardinality of X is denoted by $|X|$. The algebraic closure of a field K is denoted by $\tilde{K}$. Formulas are first order formulas in the language of rings.

1. Varieties over subfields of ample fields

We start with a well known lemma from linear algebra:

Lemma 2. *Let V be a vector space over an infinite field K. Suppose $V = \bigcup_{i\in I} W_i$ is a union of proper linear subspaces W_i of V with $\sup_{i\in I} \dim_K(W_i) < \infty$. Then $|I| \geq |K|$.*

Proof. If $\dim_K(V) = \infty$, replace V by any subspace $V' \subseteq V$ with $\sup_{i\in I} \dim_K(W_i) < \dim_K(V') < \infty$, and W_i by $W_i \cap V'$ to assume without loss of generality that $\dim_K(V) < \infty$. Then proceed by induction on $\dim_K(V)$:

The case $\dim_K(V) = 1$ clearly cannot occur. If $\dim_K(V) > 1$ and $|I| < |K|$, then there is a subspace W of V of codimension 1 such that $W \neq W_i$ for all i. But then $W = \bigcup_{i\in I}(W_i \cap W)$ and each $W_i \cap W$ is a proper subspace of W, contradicting the induction hypothesis. □

The next lemma is a refinement of the method used in [Koe02].

Lemma 3. *Let K be an ample field, $f \in K[X, Y]$ an absolutely irreducible polynomial, and $y_0 \in K$ such that $f(0, y_0) = 0$ and $\frac{\partial f}{\partial Y}(0, y_0) \neq$*

0. *Denote by C the affine curve $f(x,y)=0$, and by π the projection $(x,y)\mapsto x$. Then for every proper infinite subfield K_0 of K,*

$$|\pi(C(K))\smallsetminus K_0|\geq |K_0|.$$

Proof. Let $c\in K$. Define $x_1'=ct, x_2'=t\in K((t))$ and $g_i(Y)=f(x_i',Y)\in K[[t]][Y]$, $i=1,2$. Then g_i modulo t has the simple zero y_0. By Hensel's lemma, there are $y_1',y_2'\in K((t))$ such that $f(x_1',y_1')=f(x_2',y_2')=0$. Since K is ample, it is existentially closed in $K((t))$, cf. [Pop96, Proposition 1.1]. Therefore we can find $x_1,x_2,y_1,y_2\in K$ such that $f(x_1,y_1)=f(x_2,y_2)=0$, $x_2\neq 0$, and $c=x_1/x_2$. Since c was arbitrary, we have shown:

$$K=\left\{\frac{x_1}{x_2} : (x_1,y_1),(x_2,y_2)\in C(K), x_2\neq 0\right\}.$$

For simplicity, write $A=\{P\in C(K):\pi P\in K_0\}$, $B=\{P\in C(K):\pi P\notin K_0\}$, $A'=A\smallsetminus\pi^{-1}(0)$. Then $C(K)=A\cup B$ and $C(K)\smallsetminus\pi^{-1}(0)=A'\cup B$, so

$$\begin{aligned}
K &= \bigcup_{\substack{P\in C(K)\\ Q\in C(K)\smallsetminus\pi^{-1}(0)}}\{\frac{\pi P}{\pi Q}\}\\
&= \bigcup_{\substack{P\in A\\ Q\in A'}}\{\frac{\pi P}{\pi Q}\}\cup\bigcup_{\substack{P\in A\\ Q\in B}}\{\frac{\pi P}{\pi Q}\}\cup\bigcup_{\substack{P\in B\\ Q\in A'}}\{\frac{\pi P}{\pi Q}\}\cup\bigcup_{\substack{P\in B\\ Q\in B}}\{\frac{\pi P}{\pi Q}\}\\
&\subseteq K_0\cup\bigcup_{Q\in B}K_0\cdot\frac{1}{\pi Q}\cup\bigcup_{P\in B}K_0\cdot\pi P\cup\bigcup_{\substack{P\in B\\ Q\in B}}K_0\cdot\frac{\pi P}{\pi Q}
\end{aligned}$$

Thus, K is covered by at most $1+2\cdot|B|+|B|^2$ K_0-subspaces of dimension 1. Since $[K:K_0]\geq 2$, these are proper subspaces and Lemma 2 implies that $1+2\cdot|B|+|B|^2\geq|K_0|$, so $|B|\geq|K_0|$. Since the fibers of $\pi|_C$ are finite, this implies $|\pi(C(K))\smallsetminus K_0|=|B|\geq|K_0|$. $\square$

Proof of Theorem 1. The characterization of separating elements in separable function fields of one variable shows that we can replace φ by one of its coordinate functions and assume that $C'=\mathbb{A}^1$, i.e. φ is a separable rational function on C. Then it suffices to prove that $|\varphi(C(K))\smallsetminus K_0|=|K|$.

Since K is ample and C has a simple K-rational point, $C(K)$ is infinite. If K_0 is finite and K algebraic over K_0, then, since φ has finite fibers, $|\varphi(C(K))\smallsetminus K_0|=\aleph_0=|K|$. So assume without loss of generality that $|K_0|=\aleph_0=|K|$ or K is transcendental over K_0. Since a purely transcendental extension of a field is never ample, we may adjoin a transcendental basis of $K|K_0$ to K_0 and assume that $|K_0|=|K|$.

Since φ is separable, there is a plane affine curve $D\subseteq\mathbb{A}^2$ defined by an absolutely irreducible polynomial $f\in K[X,Y]$, separable in Y, and

a K-birational map $\eta : C \to D$ such that, with π the projection on the first coordinate, $\pi \circ \eta = \varphi$. Since η is birational, there are cofinite subsets C_0 of C and D_0 of D such that φ is defined on C_0 and η maps C_0 bijectively onto D_0 with inverse η^{-1}. In particular, $\eta(C_0(K)) = D_0(K)$ and $\varphi(C_0(K)) = \pi(D_0(K))$.

Since $C(K)$ is infinite, also $D(K)$ is infinite. Choose $P_1 = (x_1, y_1) \in D(K)$ with $\frac{\partial f}{\partial Y}(x_1, y_1) \neq 0$ and denote the curve defined by $f_1(X, Y) = f(X + x_1, Y) \in K[X, Y]$ by D_1. By Lemma 3 applied to f_1, y_1, K_0 we get that $|\pi(D_1(K)) \smallsetminus K_0| \geq |K_0| = |K|$. Thus in particular $|\pi(D(K))| = |\pi(D_1(K))| = |K|$.

Assume that $|\pi(D(K)) \smallsetminus K_0| < |K|$. Then there are infinitely many $P = (x, y) \in D(K)$ with $x = \pi P \in K_0$. Choose $P_2 = (x_2, y_2) \in D(K)$ with $x_2 = \pi P_2 \in K_0$ and $\frac{\partial f}{\partial Y}(x_2, y_2) \neq 0$ and denote the curve defined by $f_2(X, Y) = f(X + x_2, Y) \in K[X, Y]$ by D_2. By Lemma 3 applied to f_2, y_2, K_0 we get that $|\pi(D_2(K)) \smallsetminus K_0| \geq |K_0| = |K|$. But $x_2 \in K_0$ implies $|\pi(D_2(K)) \smallsetminus K_0| = |\pi(D(K)) \smallsetminus K_0| < |K|$, a contradiction.

Thus $|\pi(D(K)) \smallsetminus K_0| = |K|$. But then also $|\varphi(C(K)) \smallsetminus K_0| = |K|$, as was to be shown. $\square$

We now give two immediate consequences of Theorem 1.

Corollary 4. *Let K be an ample field and V a positive dimensional variety defined over a subfield K' of K with a simple K'-rational point. Then for every proper subfield $K_0 \supseteq K'$ of K, $|V(K) \smallsetminus V(K_0)| = |K|$. In particular, $K = K'(V(K))$ and $|V(K)| = |K|$.*

Proof. Choose an affine curve C on V defined over K' which has a simple K'-rational point (see [KA79]). By Theorem 1, $|C(K) \smallsetminus C(K_0)| = |K|$, so $|V(K) \smallsetminus V(K_0)| = |K|$. If $K_1 = K'(V(K))$ is a proper subfield of K, then $|V(K) \smallsetminus V(K_1)| = |K|$, contradicting $V(K) = V(K_1)$. $\square$

The very last statement of this corollary is the result of Pop mentioned in the introduction. A weak consequence of Corollary 4 is used in [FP08] to study the rank of Abelian varieties over ample fields.

Corollary 5. *If E is ample and $K \subseteq E$ is a subfield with* $\mathrm{tr.deg}(E|K) \geq 1$*, then for every function field of one variable $F|K$ with a rational place, there is a K-embedding of F into E.*

Proof. Let $K_0 = \tilde{K} \cap E$ and let C be a model of $F|K$ with a simple K-rational point. By Theorem 1, $|C(E) \smallsetminus C(K_0)| = |K|$. Every point $\mathbf{x} \in C(E) \smallsetminus C(K_0)$ is a generic point of C over K, so $F \cong_K K(C) \cong_K K(\mathbf{x}) \subseteq E$. $\square$

From Corollary 5 and the Riemann-Hurwitz theorem it follows immediately that function fields are non-ample and one can deduce many examples of non-ample infinite algebraic extensions of function fields. With a higher dimensional generalization of Lemma 3, one can prove

a higher dimensional analogue of Corollary 5. We will deal with these aspects in the forthcoming note [Feh08].

2. Definability in ample fields

For many of the 'classical' ample fields K, a lot is known about definable infinite subsets $X \subseteq K$. For example, if K is algebraically closed, then X is cofinite. If K is real closed then X is a finite union of intervals. If K is p-adically closed, then X contains a non-empty open subset[1]. In particular, in each of these cases K has no definable proper infinite subfields. The same is true for perfect PAC fields[2], and a recent result in [JK08] shows that it is also true in the case that K is Henselian of characteristic zero. We now show that for the huge class of perfect ample fields, there are at least no *existentially* definable proper infinite subfields.

Note that it is *not true* that an ample field of characteristic zero has no definable proper subfields (see Example 9 below), so our result on *existentially* definable subfields is in some sense the best we can expect for the class of ample fields. Furthermore note that the corresponding statement for *non-ample* fields is *not true* in general: If $F|K$ is a function field and K is algebraically closed, say, then K *is* existentially definable in F (see [Dur86], [Koe02]).

Lemma 6. *Let K be a perfect ample field and $K_0 \subseteq K$ a subfield. Let $X \subseteq K$ be existentially K_0-definable such that $X \not\subseteq \tilde{K}_0$. Then $|X \smallsetminus K_1| = |K|$ for every proper subfield K_1 of K.*

Proof. Let $\varphi(x_0)$ be an existential formula that defines X. Without loss of generality assume that $\varphi(x_0)$ is of the form

$$(\exists x_1, \dots, x_n)(\bigwedge_{i=1}^{k} f_i(x_0, x_1, \dots, x_n) = 0),$$

where $f_1, \dots, f_k \in K_0[X_0, \dots, X_n]$.

Since $X \not\subseteq \tilde{K}_0$, there are $x_0 \in X$ and $x_1, \dots, x_n \in K$ such that with $\mathbf{x} = (x_0, \dots, x_n)$, $\bigwedge_{i=1}^k f_i(\mathbf{x}) = 0$ and $\mathrm{tr.deg}(K_0(x_0)|K_0) = 1$. Let $F = K_0(\mathbf{x})$ and adjoin a transcendence base of $F|K_0(x_0)$ to K_0 to assume without loss of generality that $\mathrm{tr.deg}(F|K_0) = 1$. Then replace K_0 by $\tilde{K}_0 \cap K$, so that K_0 is perfect and $F|K_0$ is a separable function field of one variable.

If $\mathrm{char}(K) = p > 0$, let F' be the separable closure of $K_0(x_0)$ in F. Then $F' = F^{p^r} K_0 = K_0(x_0, x_1^{p^r}, \dots, x_n^{p^r})$ for some $r \geq 0$. Define $f_i'(X_0, \dots, X_n) = f_i^{p^r}(X_0, X_1^{p^{-r}}, \dots, X_n^{p^{-r}}) \in K_0[X_0, \dots, X_n]$, $\varphi'(x_0)$ as $(\exists x_1, \dots, x_n)(\bigwedge_{i=1}^k f_i'(x_0, x_1, \dots, x_n) = 0)$, and $\mathbf{x}' = (x_0, x_1^{p^r}, \dots, x_n^{p^r})$. Then $\bigwedge_{i=1}^k f_i'(\mathbf{x}') = 0$ and $K_0(\mathbf{x}') = F'$. Since K is perfect, the subset

[1] see [Mac76]

[2] as follows from [CvdDM92, Proposition 5.3]

defined by φ' is exactly X. Thus replace φ by φ', $\mathbf{x}$ by $\mathbf{x}'$ and F by F' to assume without loss of generality that $F|K_0(x_0)$ is separable.

Let C be the locus of $\mathbf{x}$ over K_0. That is, $C \subseteq \mathbb{A}^{n+1}$ is an affine curve defined over K_0 with $F \cong_{K_0} K_0(C)$, and $\mathbf{x}$ is a simple F-rational point of C. If $\mathbf{y} \in C(K)$ is another point of C, then there is a K_0-specialization $\eta_{\mathbf{y}} : K_0[\mathbf{x}] \to_{K_0} K_0[\mathbf{y}]$. In particular, $0 = \eta_{\mathbf{y}}(f_i(\mathbf{x})) = f_i(\mathbf{y})$ for all i, i.e. $y_0 \in X$. This means that if $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^1$ is the projection on the first coordinate, then $\pi(C(K)) \subseteq X$.

But $\pi|_C$ is separable since $F|K_0(x_0)$ is separable, so $|\pi(C(K)) \smallsetminus K_1| = |K|$ by Theorem 1. Thus $|X \smallsetminus K_1| \geq |\pi(C(K)) \smallsetminus K_1| = |K|$. $\square$

Corollary 7. *Let K be a perfect ample field and let $X \subseteq K$ be existentially K-definable and infinite. Then $|X \smallsetminus K_1| = |K|$ for every proper subfield K_1 of K.*

Proof. Let φ be an existential formula that defines X, and let $K_0 \subseteq K$ be the field generated by the parameters used in φ. Let $K^* \supseteq K$ be an $\aleph_1$-saturated ultrapower of K. Then the subset $X^* \subseteq K^*$ defined by φ in K^* is uncountable. Since K_0 is finitely generated, $X^* \not\subseteq \tilde{K}_0$. Thus by Lemma 6, $|X^* \smallsetminus K_1^*| = |K^*|$ for every proper subfield K_1^* of K^*.

Let K_1 be any proper subfield of K. Then the corresponding ultrapower K_1^* is a proper subfield of K^*, thus $|X^* \smallsetminus K_1^*| = |K^*|$, which implies $|X \smallsetminus K_1| = \infty$. So if K is countable, $|X \smallsetminus K_1| = \aleph_0 = |K|$ as was to be shown. If K is uncountable, then $\tilde{K}_0 \cap K$ is a proper subfield of K, so $|X \smallsetminus (\tilde{K}_0 \cap K)| = \infty$. In particular $X \not\subseteq \tilde{K}_0$. Thus by Lemma 6, $|X \smallsetminus K_1| = |K|$. $\square$

Corollary 8. *A perfect ample field K has no existentially K-definable proper infinite subfields.* $\square$

The following example shows that Corollary 8 does not hold for arbitrary K-definable subfields (rather than existentially K-definable subfields):

Example 9. *The prime field $\mathbb{Q}$ is a $\emptyset$-definable proper subfield of the ample field $\mathbb{Q}((X,Y))$.*

Proof. As noted in [Pop08], $K((X,Y))$ is ample for any field K. But the ring $K[[X,Y]]$ is definable in $K((X,Y))$, see [JL89, Theorem 3.34], and if $\mathrm{char}(K) = 0$, then $\mathbb{N}$ is definable in $K[[X,Y]]$, see [Del81, Théorème 2.1]. $\square$

Note that the results of this section are closely related to [JK08]. There, a field K is called *very slim*, if in all fields elementary equivalent to K, 'the model theoretic algebraic closure in K coincides with the relative field theoretic algebraic closure', cf. [JK08, Definition 1.1].

Since a very slim field has no definable proper infinite subfields, see [JK08, Proposition 4.1], Example 9 gives a positive answer to (1) of [JK08, Question 8]: Perfect ample fields that are not very slim do

exist. In particular, [JK08, Theorem 5.5], which states that Henselian fields of characteristic zero are very slim, does not generalise to the class of ample fields of characteristic zero.

Since a model complete field is perfect and all its definable subsets are existentially definable, Lemma 6 immediately gives a new proof for [JK08, Theorem 5.4]: Every model complete ample field is very slim.

Acknowledgements

The author would like to thank Lior Bary-Soroker and Moshe Jarden for helpful comments and encouragement, and Elad Paran for contributing to the introduction. This work was supported by the European Commission under contract MRTN-CT-2006-035495.

References

[CvdDM92] Zoé Chatzidakis, Lou van den Dries, and Angus Macintyre. Definable sets over finite fields. *Journal für die reine und angewandte Mathematik*, 427:107–135, 1992.

[DD99] Pierre Dèbes and Bruno Deschamps. The regular inverse galois problem over large fields. *Geometric Galois actions, London Math. Soc., Lec. Note Ser.*, 2(8):119–138, 1999.

[Del81] Françoise Delon. Indécidabilité de la théorie des anneaux de séries formelles à plusiers indéterminées. *Fundamenta Mathmaticae*, 112:215–229, 1981.

[Dur86] Jean-Louis Duret. Sur la théorie élémentaire des corps de fonctions. *Journal of Symbolic Logic*, 51(4):948–956, 1986.

[Feh08] Arno Fehm. Subfields of ample fields II. Embeddings into ample fields. 2008. Manuscript.

[FP08] Arno Fehm and Sebastian Petersen. On the rank of abelian varieties over ample fields. Manuscript, 2008.

[Har06] David Harbater. On function fields with free absolute galois groups. 2006. arXiv:math/0608584v1 [math.AG].

[HS05] David Harbater and Katherine F. Stevenson. Local galois theory in dimension two. *Advances in Mathematics*, 198:623–653, 2005.

[JK08] Markus Junker and Jochen Koenigsmann. Schlanke Körper (Slim Fields). Manuscript, 2008.

[JL89] Christian U. Jensen and Helmut Lenzing. *Model Theoretic Algebra*. Algebra, Logic and Applications 2. Gordon and Breach, 1989.

[KA79] Steven L. Kleiman and Allen B. Altman. Bertini theorems for hypersurfaces sections containing a subscheme. *Communications in Algebra*, 7(8):775–790, 1979.

[Kob06] Emi Kobayashi. A remark on the Mordell-Weil rank of elliptic curves over the maximal abelian extension of the rational number field. *Tokyo Journal of Mathematics*, 29(2):295–300, 2006.

[Koe02] Jochen Koenigsmann. Defining transcendentals in function fields. *Journal of Symbolic Logic*, 67(3):947–956, 2002.

[Kol99] János Kollár. Rationally connected varieties over local fields. *Annals of Mathematics*, 150:357–367, 1999.

[LR08] Álvaro Lozano-Robledo. Ranks of abelian varieties over infinite extensions of the rationals. to appear in Manuscripta Mathematica, 2008.

[Mac76] Angus Macintyre. On definable subsets of p-adic fields. *Journal of Symbolic Logic*, 41(3):605–610, 1976.

[MB01] Laurent Moret-Bailly. R-équivalence simultanée de torseurs: un complément à l'article de P. Gille. *Journal of Number Theory*, 91(2):293–296, 2001.

[Pop96] Florian Pop. Embedding problems over large fields. *Annals of Mathematics*, 144:1–34, 1996.

[Pop08] Florian Pop. Henselian implies large. Manuscript, 2008.

[PP07] Bjorn Poonen and Florian Pop. First-order definitions in function fields over anti-mordellic fields. In Chatzidakis, Macpherson, Pillay, and Wilkie, editors, *Model Theory with applications to algebra and analysis*. Cambridge University Press, 2007.